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# THE $\overline{\partial}$ PROBLEM ON DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES WITH APPLICATIONS

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Abstract. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ such that $\Omega$ has piecewise smooth boudnary. We discuss the solvability of the Cauchy-Riemann equation

$$\overline{\partial} u = \alpha \quad \text{in} \quad \Omega \tag{0.1}$$

where $\alpha$ is a smooth $\overline{\partial}$-closed $(p,q)$ form with coefficients $C^\infty$ up to the bundary of $\Omega$, $0 \leq p \leq n$ and $1 \leq q \leq n$. In particular, Equation (0.1) is solvable with $u$ smooth up to the boundary (for appropriate degree $q$) if $\Omega$ satisfies one of the following conditions:

i) $\Omega$ is the transversal intersection of bounded smooth pseudoconvex domains.

ii) $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ where $\Omega_2$ is the union of bounded smooth pseudoconvex domains and $\Omega_1$ is a pseudoconvex convex domain with a piecewise smooth boundary.

iii) $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ where $\Omega_2$ is the intersection of bounded smooth pseudoconvex domains and $\Omega_1$ is a pseudoconvex domain with a piecewise smooth boundary.

The solvability of Equation (0.1) with solutions smooth up to the boundary can be used to obtain the local solvability for $\overline{\partial}_b$ on domains with piecewise smooth boundaries in a pseudoconvex manifold.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ such that $\Omega$ has a piecewise smooth boundary. In this paper we study the solvability of the Cauchy-Riemann equation

$$\overline{\partial} u = \alpha \quad \text{in} \quad \Omega, \tag{0.1}$$

where $\alpha$ is a smooth $\overline{\partial}$-closed $(p,q)$ form with coefficients $C^\infty$ up to the boundary of $\Omega$, $0 \leq p \leq n$ and $1 \leq q \leq n$. In particular, we prove that Equation (0.1) is solvable with $u$ smooth up to the boundary (for appropriate degree $q$) if $\Omega$ satisfies one of the following conditions:

i) $\Omega$ is the transversal intersection of bounded smooth pseudoconvex domains.

ii) $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where $\Omega_2$ is the union of bounded smooth pseudoconvex domains and $\Omega_1$ is a pseudoconvex domain with a piecewise smooth boundary.

iii) $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where $\Omega_2$ is the intersection of bounded smooth pseudoconvex domains and $\Omega_1$ is a pseudoconvex domain with a piecewise smooth boundary.

The boundary regularity problem for $\overline{\partial}$ has been studied extensively by two very different methods: by $L^2$ a priori estimates and by the integral kernel approach.

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The $L^2$ method was first used by Kohn [15] in studying the boundary regularity of the $\overline{\partial}$ equation when $\Omega$ is smooth and strictly pseudoconvex. He established the existence and regularity for the $\overline{\partial}$-Neumann operator and obtained subellipitic estimates in the Sobolev spaces for the solutions. Regularity results for the $\overline{\partial}$-Neumann operator have also been obtained for a wide class of other pseudoconvex domains with smooth boundaries (see Kohn [17], Catlin [3] and Boas-Straube [2]). Thanks to Hörmander's (see [13]) $L^2$ existence theorem for $\overline{\partial}$, the $\overline{\partial}$-Neumann operator is known to exist for any bounded pseudoconvex domain. On the other hand, recent results have shown that the $\overline{\partial}$-Neumann operator can be irregular on certain pseudoconvex domains with $C^\infty$ boundaries (see Barrett [1] and Christ [5]). It is also known that the $\overline{\partial}$-Neumann operator on domains with nonsmooth boundaries does not preserve $C^\infty$ smoothness (see Michel-Shaw [24]), even for domains with piecewise strictly pseudoconvex boundaries. When $\Omega$ is pseudoconvex with $C^\infty$ boundary, the $\overline{\partial}$ equation was studied in Kohn [16] using the weighted $\overline{\partial}$-Neumann operator. Thus the method of $L^2$ a priori estimates for the (weighted) $\overline{\partial}$-Neumann operator has yielded many important results on the local and global boundary regularity of Equation (0.1) when the domain is pseudoconvex with $C^\infty$ boundary. On the other hand, this method is not easily adapted to nonsmooth domains.

The integral formula approach was pioneered by Henkin [9] and Grauert-Lieb [8] for strictly pseudoconvex domains. They obtained uniform and Hölder estimates for the solution of $\overline{\partial}$ on such domains. The integral formula was extended subsequently to analytic polyhedra (see Henkin [10]) and piecewise strictly pseudoconvex domains (see Polyakov [27], [28] and Range-Siu [30]), where uniform and Hölder estimates were obtained. The $C^k$ estimates for $\overline{\partial}$ on strictly pseudoconvex domains were studied in Lieb-Range [18] and by Michel [20] in the piecewise smooth case.

In this paper, we combine the two approaches to study the boundary regularity of $\overline{\partial}$ on domains with piecewise smooth boundaries. We first use the $L^2$ method to construct certain barrier functions which satisfy a certain growth condition for smooth pseudoconvex domains, using the weighted $\overline{\partial}$-Neumann operator. Then we use the barriers to construct kernels for the homotopy formula on piecewise smooth domains. Since the main ingredient in the kernel approach is the Stokes' theorem, one can extend the construction for smooth domains to piecewise smooth domains along the lines of the previous work in the strictly pseudoconvex case. As a result, we have obtained solutions of $\overline{\partial}$ smooth up to the boundary for the transversal intersection of bounded smooth pseudoconvex domains. We should mention that when $\Omega$ is pseudoconvex with a special Stein neighborhood basis, the $C^\infty$ regularity was obtained by Dufresnoy [7] (see also Chaumat-Chollet [4]).

We also construct kernels for the annuli between two piecewise smooth domains. When $\Omega$ is an annulus between two smooth bounded pseudoconvex domains with $C^\infty$ boundaries, such results have been obtained in Shaw [31]. It was first proved by Hortmann [14] that one can construct a homotopy formula for the annulus between two strictly pseudoconvex domains. Recently Michel-Shaw [25] have extended this result to the case when $\Omega = \Omega_1 \setminus \Omega_2$ and the boundary of $\Omega_2$ is pseudoconvex with only $C^2$ boundary. When $\Omega_1$, $\Omega_2$ have piecewise smooth strongly pseudoconvex boundaries, Equation (0.1) was studied in Michel-Perotti [22], [23].

The $\overline{\partial}$ problem on piecewise smooth domains is not only interesting in itself, it also arises from the local solvability of tangential Cauchy-Riemann equations. Let $\omega$ be an open subset of the boundary $M$ of a bounded smooth pseudoconvex domain in $\mathbb{C}^n$. We consider the equation

$$\overline{\partial}_b u = \alpha \quad \text{in} \quad \omega, \tag{0.2}$$

where $\alpha$ is a smooth $\overline{\partial}_b$-closed $(p,q)$ form on $\overline{\omega}$, $1 \le q \le n-3$. We show that when the boundary $\partial\omega$ lies locally in the transversal intersection of $M$ with $k$ Levi-flat hypersurfaces, then one can find a smooth solution $u$ satisfying (0.2) for $1 \le q \le n-k-2$, provided these $k$ hypersurfaces satisfy some global conditions. This is the first result for solvability of $\overline{\partial}_b$ on pseudo-convex hypersurfaces with piecewise smooth boundaries. Previous results (cf. Henkin [11], Shaw [32], [33], Michel-Shaw [26]) all require that the boundary be smooth and that $\partial\omega$ lie in some Levi-flat hypersurface. Thus our result is much more general even for the smooth boundary case.

The proof of local solvability for Equation (0.2) depends on solving the $\overline{\partial}$-equation on piecewise smooth domains with compact support, which is equivalent to solving $\overline{\partial}$ on an annulus with piecewise smooth boundary such that the solutions are smooth up to the boundary (see Theorem 3 and Corollary 3.1). At the end of this paper we shall give examples to show that our results are sharp in the appropriate sense. Explicit kernels for $\overline{\partial}_b$ on a domain in a convex hypersurface with piecewise smooth boundary have been constructed recently by Vassiliadou [36].

The plan of this paper is as follows: In section I we derive a homotopy formula for $\overline{\partial}$ on a domain with a piecewise smooth boundary using the barrier functions constructed in Michel [21]. In section II we construct a homotopy formula on an annulus such that $\Omega_2$ is the union of finitely many smooth pseudoconvex domains. The proof depends on the barrier functions constructed recently in Michel-Shaw [25]. We then use induction to construct a solution for Equation (0.1) when $\Omega_2$ is the transversal intersection of finitely many smooth pseudoconvex domains. In section III we prove the solvability of Equation (0.2) with regularity up to the boundary on a domain $\omega$, using results proved in sections I and II.

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## I. Homotopy formulas for $\overline{\partial}$ on piecewise smooth pseudoconvex domains

Let $D_i$, $i = 1, \cdots, k$, be bounded pseudoconvex domains in $\mathbb{C}^n$ with $C^\infty$ boundary $\partial D_i$. Let $\rho_i$ be a $C^\infty$ defining function for $D_i$, i.e., $D_i = \{z \in \mathbb{C}^n \mid \rho_i(z) < 0\}$ and $\nabla\rho_i \neq 0$ on $\partial D_i$. Let $D = \bigcap_{i=1}^k D_i$ be the transversal intersection of the $D_i$'s. We shall call $D$ a piecewise smooth pseudoconvex domain in $\mathbb{C}^n$ if

$$d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_\ell} \neq 0$$

on $\{z \in \partial D | \rho_{i_1} = \cdots = \rho_{i_\ell} = 0\}$ for every $1 \le i_1 < i_2 < \cdots < i_\ell \le k$. In this section we construct a homotopy formula for $\overline{\partial}$ on such $D$. When $D_i$ is strictly pseudoconvex, the homotopy formulas were constructed in Polyakov [28] and Range-Siu [30]. We refer the readers to these papers and to the books of Henkin-Leiterer [12] and Range [29] for the details of the construction of the kernel.

Let $\mathcal{U}_i$ be an open neighborhood of $D_i$. It follows from Michel [21] that for each $D_i$ there exist an increasing sequence $0 < t_0 < \cdots$ and a $C^3$ mapping $w^{(i)}$ :

$D_i \times (\mathcal{U}_i \setminus \overline{D}_i) \to \mathbb{C}^n$ satisfying

$$\overline{\partial}_z \omega^{(i)}(z,\zeta) = 0, \quad \sum_{\nu=1}^{n} (\zeta_\nu - z_\nu)\omega_\nu^{(i)}(z,\zeta) = 1 \tag{1.1}$$

and the estimates

$$| \, \omega^{(i)}(\cdot,\zeta) \, |_s + | \, \nabla_\zeta \omega^{(i)}(\cdot,\zeta) \, |_s \leq \frac{C(s)}{| \, \rho_i(\zeta) \, |^{t_s}} \text{ for every } s \in \mathbb{N} \cup \{0\}, \tag{1.2}$$

where $\zeta \in \mathcal{U}_i \setminus \overline{D}_i$, $| \ \ |_s$ denotes the $C^s(\overline{D})$ norm and $C(s)$ does not depend on $\zeta$. (1.2) also implies that $\omega^{(i)}(\cdot,\zeta)$ is in $(C^\infty(\overline{D}_i))^n$.

We define some special Cauchy-Fantappiè forms with $\omega_\nu^{(i)}$.

Let

$$\omega_\nu^{(0)}(z,\zeta) = \frac{\overline{\zeta}_\nu - \overline{z}_\nu}{| \, \zeta - z \, |^2}, \quad 1 \leq \nu \leq n,$$

and

$$\omega_\nu(z,\zeta,\lambda) = \sum_{j=0}^{n} \lambda_j \omega_\nu^{(j)}(z,\zeta),$$

wherever this is defined.

Let $\omega$ be the column vector $\begin{pmatrix} \omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega_n \end{pmatrix}$. For $0 \leq q \leq n-1$, we define

$$\Omega_{n,q} = (-1)^q \binom{n-1}{q} \det(\omega, \underbrace{\overline{\partial}_z \omega, \cdots, \overline{\partial}_z \omega}_{q \text{ times}}, \underbrace{\overline{\partial}_{\zeta,\lambda} \omega, \cdots, \overline{\partial}_{\zeta,\lambda} \omega}_{n-q-1 \text{ times}}) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

Set $\Omega_{n,n} = \Omega_{n,-1} = 0$. It follows that

$$\overline{\partial}_{\zeta,\lambda} \ \Omega_{n,q} = (-1)^q \ \overline{\partial}_z \Omega_{n,q-1}$$

and

$$d_{\zeta,\lambda}(U \wedge \Omega_{n,q}) = U \wedge \overline{\partial}_z \Omega_{n,q-1} + \overline{\partial}_\zeta U \wedge \Omega_{n,q} \quad \text{for} \ \ U \in C^1_{(0,q)}(\overline{D}).$$

For every ordered subset $I = \{i_1, \cdots, i_\ell\}$ of $\{1, \cdots, k\}$ we define

$$S_I = \{x \in \partial D \mid \rho_i(x) = 0 \text{ for } i \in I\}$$

and choose the orientation on $S_I$ such that the orientation is skew symmetric in the components of $I$ and the following equations hold when $D$ is given the natural orientation:

$$\partial D = \sum_{j=1}^{k} S_j,$$

$$\partial S_I = \sum_{j=1}^{k} S_{Ij}.$$

Let

$$\triangle = \{\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_k) \in R^{k+1} \mid \lambda_i \geq 0, \lambda_0 + \cdots + \lambda_k = 1\}.$$

For each ordered subset $J = \{j_1, \cdots, j_m\}$ of $\{1, \cdots, k\}$, we define

$$\triangle_J = \{\lambda \in \triangle \mid \sum_{j \in J} \lambda_j = 1\}.$$

The orientation of each $\triangle_J$ is chosen so that

$$\partial \triangle_J = \sum_{\nu=1}^{m} (-1)^{\nu+1} \triangle_{j_1 \cdots \widehat{j_\nu} \cdots j_m},$$

where $\widehat{j_\nu}$ means that $j_\nu$ is omitted.

Let $D_0$ be a small neighborhood of $\overline{D}$ such that for some small $\epsilon_0 > 0$, $D_0 = \{z \in \mathbb{C}^n \mid \rho_i(z) < \epsilon_0,\ i = 1, \cdots, k\}$. We define

$$S_I^0 = \{x \in \partial D_0 \mid \rho_i(x) = \epsilon_0 \ \text{ for } i \in I\}$$

and

$$R_I = \{z \in \overline{D}_0 \mid 0 \le \rho_{i_1}(z) = \cdots = \rho_{i_\ell}(z) \le \epsilon_0,\ \rho_j(z) \le \rho_{i_1}(z) \ \text{ for } j \notin I\}.$$

We also require that the orientation on $R_I$ be skew symmetric in the components of $I$ and $R = \sum_{i=1}^{k} R_i = \overline{D}_0 \setminus D$. It follows (see Range-Siu [30]) that

$$\partial R_I = \sum_{j=1}^{k} R_{Ij} - S_I + S_I^0$$

and

$$\begin{aligned} \partial\Big(\sum_I (-1)^{|I|} R_I \times \triangle_{0I}\Big) &= \sum_I R_I \times \triangle_I - R \times \triangle_0 \\ &\quad + \sum_I (-1)^{|I|} S_I \times \triangle_{0I} - \sum_I (-1)^{|I|} S_I^0 \times \triangle_{0I}, \end{aligned} \tag{1.3}$$

where the summation is over all ordered increasing subsets of $\{1, \cdots, k\}$.

Following the Bochner-Martinelli-Koppelman formula, for any $u \in C^1_{(0,q)}(\overline{D})$ we have

$$u(z) = C_n\Big( \int_{\partial D \times \triangle_0} u \wedge \Omega_{n,q} - \int_{D \times \triangle_0} \overline{\partial}_\zeta u \wedge \Omega_{n,q} - \overline{\partial}_z \int_{D \times \triangle_0} u \wedge \Omega_{n,q-1} \Big) \tag{1.4}$$

for $z \in D$, where $C_n = (-1)^{n(n-1)/2}/(2\pi i)^n$. For a detailed proof of (1.4), see e.g. Lemma 2.4 in Range-Siu [30] or Theorem 1.10 in Range [29].

For any $V \in C^\infty_{(0,q+1)}(\overline{D_0 \setminus D})$ such that $V$ vanishes on $\partial D$ and $\partial D_0$, we apply Stokes' theorem to the form $d_{\zeta,\lambda}(V \wedge \Omega_{n,q})$ and the $2n$-chain $(\sum_I (-1)^{|I|} R_I \times \triangle_{0I})$, where $\sum_I$ is over all increasing ordered subsets of $\{1, 2, \cdots, k\}$. We use (1.3) and the formula

$$d_{\zeta,\lambda}(V \wedge \Omega_{n,q}) = -V \wedge \overline{\partial}_z \Omega_{n,q-1} + \overline{\partial}_\zeta V \wedge \Omega_{n,q}$$

to obtain

$$\begin{aligned} &- \int_{\sum_I (-1)^{|I|} R_I \times \triangle_{0I}} V \wedge \overline{\partial}_z \Omega_{n,q-1} + \int_{\sum_I (-1)^{|I|} R_I \times \triangle_{0I}} \overline{\partial}_\zeta V \wedge \Omega_{n,q} \\ &\qquad = \int_{\sum_I R_I \times \triangle_I} V \wedge \Omega_{n,q} - \int_{R \times \triangle_0} V \wedge \Omega_{n,q}. \end{aligned} \tag{1.5}$$

Since the boundary of $D$ is Lipschitz, for any $\alpha \in C^\infty(\overline{D})$ there exists an extension $E\alpha \in C^\infty(\mathbb{C}^n)$, and $E\alpha$ vanishes outside a neighborhood of $\overline{D}$ (see e.g. Section 6.3 in Stein [35]). Furthermore, $E$ is bounded from $C^{k,\epsilon}(\overline{D})$ to $C^{k,\epsilon}(\mathbb{C}^n)$, where $0 < \epsilon < 1$. Let $f \in C^\infty_{(0,q)}(\overline{D})$ and apply $E$ to $f$ componentwise, so that supp $Ef \subset D_0$.

Applying the Bochner-Martinelli-Koppelman formula to $Ef$ on the region $R$, we have, for $z \in D$,

$$-\int\limits_{R\times\triangle_0} \overline{\partial} Ef \wedge \Omega_{n,q} = \overline{\partial}_z \int\limits_{R\times\triangle_0} Ef \wedge \Omega_{n,q-1} + \int\limits_{\partial D\times\triangle_0} f \wedge \Omega_{n,q}.$$

If $V = \overline{\partial} Ef - E\overline{\partial} f$, we have, using (1.4) for $z \in D$,

$$\begin{aligned} -\int\limits_{R\times\triangle_0} V \wedge \Omega_{n,q} &= -\int\limits_{R\times\triangle_0} \overline{\partial} Ef \wedge \Omega_{n,q} + \int\limits_{R\times\triangle_0} E\overline{\partial} f \wedge \Omega_{n,q} \\ &= \overline{\partial}_z \int\limits_{R\times\triangle_0} Ef \wedge \Omega_{n,q-1} + \int\limits_{\partial D\times\triangle_0} f \wedge \Omega_{n,q} + \int\limits_{R\times\triangle_0} E\overline{\partial} f \wedge \Omega_{n,q} \\ &= \overline{\partial}_z \int\limits_{D_0\times\triangle_0} Ef \wedge \Omega_{n,q-1} + \int\limits_{D_0\times\triangle_0} E\overline{\partial} f \wedge \Omega_{n,q} + C_n^{-1} f(z). \end{aligned} \tag{1.6}$$

Since $V$ vanishes on $\partial D$ and $\partial D_0$, we can apply (1.5) to obtain

$$\begin{aligned} C_n^{-1} f(z) = -\overline{\partial}_z & \int\limits_{\sum_I (-1)^{|I|} R_I\times\triangle_{0I}} V \wedge \Omega_{n,q-1} \\ & + \int\limits_{\sum_I (-1)^{|I|} R_I\times\triangle_{0I}} \overline{\partial}_\zeta V \wedge \Omega_{n,q} - \int\limits_{\sum_I R_I\times\triangle_I} V \wedge \Omega_{n,q} \\ & - \overline{\partial}_z \int\limits_{D_0\times\triangle_0} Ef \wedge \Omega_{n,q-1} - \int\limits_{D_0\times\triangle_0} E\overline{\partial} f \wedge \Omega_{n,q}. \end{aligned} \tag{1.7}$$

Since each $\omega_\nu^{(j)}(\zeta, z)$ is holomorphic in $z$ for $1 \le j \le k$ and $1 \le \nu \le n$, we have

$$\Omega_{n,q} = 0 \qquad \text{on } R_I \times \triangle_I \quad \text{for } q > 0.$$

Thus we define

$$T_0 f = -C_n \sum_I \int\limits_{R_I\times\triangle_I} (\overline{\partial} Ef - E\overline{\partial} f) \wedge \Omega_{n,0}, \tag{1.8}$$

$$\begin{aligned} T_q f = C_n \sum_I (-1)^{|I|+1} & \int\limits_{R_I\times\triangle_{0I}} (\overline{\partial} Ef - E\overline{\partial} f) \wedge \Omega_{n,q-1} \\ & - C_n \int\limits_{D_0\times\triangle_0} Ef \wedge \Omega_{n,q-1}. \end{aligned} \tag{1.9}$$

Then it follows from (1.7) that, for $z \in D$,

$$f = \overline{\partial} T_q f + T_{q+1} \overline{\partial} f \quad \text{for } f \in C^\infty_{(0,q)}(\overline{D}),\ 1 \le q \le n, \tag{1.10}$$

and

$$f = T_0 f + T_1 \overline{\partial} f \quad \text{for } f \in C^{\infty}_{(0,0)}(\overline{D}). \tag{1.11}$$

**Theorem 1.** *Let $D$ be a piecewise smooth pseudoconvex domain in $\mathbb{C}^n$. For $1 \leq q \leq n$, there exist linear operators $T_q : C^{\infty}_{(0,q)}(\overline{D}) \longrightarrow C^{\infty}_{(0,q-1)}(\overline{D})$ such that, for every $f \in C^{\infty}_{(0,q)}(\overline{D})$,*

$$f = \overline{\partial} T_q f + T_{q+1} \overline{\partial} f$$

*where we have set $T_{n+1} = 0$. When $q = 0$, there exists a linear operator $T_0 : C^{\infty}(\overline{D}) \longrightarrow A^{\infty}(\overline{D})$ such that*

$$f = T_0 f + T_1 \overline{\partial} f,$$

*where $A^{\infty}(\overline{D})$ is the subspace of holomorphic functions in $C^{\infty}(\overline{D})$.*

**Corollary 1.1.** *Let $D$ be the same as in Theorem 1. Let $\alpha \in C^{\infty}_{(0,q)}(\overline{D})$, where $1 \leq q \leq n$ and $\overline{\partial}\alpha = 0$ in $D$. There exists a $u \in C^{\infty}_{(0,q-1)}(\overline{D})$ such that $\overline{\partial} u = \alpha$ in $D$.*

*Proof.* Let $T_q$ be defined by (1.8) and (1.9). We only need to show that $T_q f \in C^{\infty}_{(0,q-1)}(\overline{D})$. It is obvious that $\int_{D_0 \times \triangle_0} Ef \wedge \Omega_{n,q-1} \in C^{\infty}_{(0,q-1)}(\overline{D})$. To show that $\int_{R_I \times \triangle_{0I}} (\overline{\partial} Ef - E\overline{\partial} f) \wedge \Omega_{n,q-1}$ is in $C^{\infty}_{(0,q-1)}(\overline{D})$, we note that $V = \overline{\partial} Ef - E\overline{\partial} f$ vanishes to infinite order on $\partial D$. Letting $d(\zeta)$ denote the distance from $\zeta$ to $\partial D$, we have

$$|V(\zeta)| \leq C_N |d(\zeta)|^N \quad \text{for any} \quad N \in \mathbb{N}. \tag{1.12}$$

Since $\partial D$ is Lipschitz, for any $\zeta \in D_0 \setminus D$ and $z \in D$ there exists a constant $C > 0$ such that

$$|\zeta - z| \geq C|d(\zeta)|, \tag{1.13}$$

where $C$ is independent of $\zeta$ and $z$. Also, for any $\zeta \in R_I$, there exists a $C > 0$ such that we have

$$|\rho_i(\zeta)| \geq C|d(\zeta)| \quad \text{for any } i \in I, \tag{1.14}$$

where $C$ is independent of $\zeta$. It follows from (1.2) and (1.13), (1.14) that for any $s \in \mathbb{N}$, there exist $C_s$, $T_s$ such that

$$|\Omega_{n,q-1}(\cdot, \zeta)|_s \leq \frac{C_s}{|d(\zeta)|^{T s}}. \tag{1.15}$$

The theorem follows from (1.12), (1.15) and differentiation under the integral sign. □

## II. Boundary regularity for $\overline{\partial}$ on piecewise smooth annuli

Let $\Omega$ be a bounded piecewise smooth pseudoconvex domain in $\mathbb{C}^n$. Let $D_i \subset\subset \Omega$, $i = 1, \cdots, k$, be such that each $D_i$ is a bounded pseudoconvex domain with $C^2$ boundary $\partial D_i$ defined by $\{\rho_i = 0\}$. We assume that $d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_\ell} \neq 0$ on $\rho_{i_1} = \cdots = \rho_{i_\ell} = 0$ for every $I = (i_1, \cdots, i_\ell)$, $1 \leq i_1 < \cdots < i_\ell \leq k$. Let

$$D = \Omega \setminus \left( \overline{\bigcup_{i=1}^{k} D_i} \right).$$

Then $D$ is the annulus between a pseudoconvex domain $\Omega$ and the union of finitely many bounded pseudoconvex domains with $C^2$ boundary. In this section we consider $\overline{\partial}$ on $D$ with solutions smooth up to the boundary.

We shall construct a homotopy formula for $\overline{\partial}$ on $D$. Since each $D_i$ has $C^2$ boundary, it follows from Diederich-Fornaess [6] that there exist a $C^2$ defining function $\tilde{\rho}_i$ and a $\nu_i \geq 1$ such that $\phi_i = -(-\tilde{\rho}_i)^{1/\nu_i}$ is a bounded strictly plurisubharmonic exhaustion function on $D_i$. Let $\mathcal{U}_i = \Omega \setminus \overline{D}_i$ and $W^i = D_i \times \mathcal{U}_i,\ i = 1, \cdots, k$. For each nonnegative integer $m$, there exists a $C^m$ map

$$w^{(i)} = (w_1^{(i)}, \cdots, w_n^{(i)}) \ : \ W^i \longrightarrow \mathbb{C}^n$$

such that a) $\omega^{(i)}(\cdot, \zeta)$ is holomorphic for every $\zeta \in \mathcal{U}_i$, and for each $(z, \zeta) \in W^i$ we have

$$\sum_{\mu=1}^{n} w_\mu^{(i)}(z,\zeta)(\zeta_\mu - z_\mu) = 1,$$

b) and there exists a constant $C(m)$ such that for all $I$, with $\mid I \mid\leq m$, for all $\zeta \in \mathcal{U}_i$ any any $z \in D_i$,

$$\mid D_\zeta^I \omega^{(i)}(z,\zeta) \mid \leq C(m) \text{ dist } (z, bD_i)^{-N_m}, \quad i = 1, \cdots, k. \tag{2.1}$$

where $N_m = (5t_m^2 + 3|I| + 1)/\nu_i + 2n$, $t_m = [2\nu \max{(4 + 3m, \frac{n-1}{5})} + 1]$ and [a] denotes the largest integer $j \leq a$. Each $w^{(i)}$ depends on $m$, though we do not indicate it in the notation. Such a barrier function $w^{(i)}$ exists for each pseudoconvex domain with $C^2$ boundary and was constructed in Theorem 1 in [25]. We set, for $z \in \mathcal{U}_i,\ \zeta \in D_i$,

$$P_\mu^{(i)}(z,\zeta) = -w_\mu^{(i)}(\zeta, z), \quad i = 1, \cdots, k,\ \mu = 1, \cdots, n.$$

Then $P^{(i)} = (P_1^{(i)}, \cdots, P_n^{(i)}) \ : \ \mathcal{U}_i \times D_i \longrightarrow \mathbb{C}^n$ is holomorphic in $\zeta \in D_i$ for each fixed $z \in \mathcal{U}_i$. We set $P_\mu^0(z,\zeta) = (\overline{\zeta}_\mu - \overline{z}_\mu)/ \mid \zeta - z \mid^2$ for $\mu = 1, \cdots, n$, and we define

$$P_\mu(z,\zeta,\lambda) = \sum_{j=0}^{k} \lambda_j P_\mu^{(j)}(z,\zeta)$$

wherever it is defined, where $\lambda_j \geq 0$ and $\lambda_0 + \lambda_1 + \cdots + \lambda_k = 1$. In particular, if $\lambda_{i_1} + \cdots + \lambda_{i_r} = 1$, $z \in D$ and $\zeta \in \bigcap_{\nu=1}^r D_{i_\nu}$, then $P_\mu$ is well defined and holomorphic in $\zeta$ if $\lambda_0 = 0$. Let $P$ be the column vector $\begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}$. For $0 \leq q \leq n-1$, we define the double differential form

$$\Omega_{n,q}^0 = (-1)^q \binom{n-1}{q} \det(P, \underbrace{\overline{\partial}_z P, \ldots, \overline{\partial}_z P}_{q}, \underbrace{\overline{\partial}_{\zeta,\lambda} P, \ldots, \overline{\partial}_{\zeta,\lambda} P}_{n-q-1}) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

which is of degree $q$ in $z$ and of degree $2n - q - 1$ in $(\zeta, \lambda)$. Set $\Omega_{n,n}^0 = \Omega_{n,-1}^0 = 0$ and $\bigcup_{i=1}^k D_i = R^0$. For each increasing index $I = (i_1, \cdots, i_\ell),\ 1 \leq \ell \leq k$, we define, for small $\epsilon_0 > 0$,

$$R_I^0 = \{ z \in \bigcap_{\nu=1}^{\ell} D_{i_\nu} \mid -\epsilon_0 \leq \rho_{i_1}(z) = \cdots = \rho_{i_\ell}(z) \leq 0,\ \rho_j(z) \geq \rho_{i_1}(z)$$
$$\text{for } j \notin I \text{ and } z \in D_j \}.$$

Similarly as before, we require that the orientation on $R_I^0$ be skew symmetric in the components of $I$, and we define

$$S_I = \{z \in \partial \bigcup_{i=1}^{k} D_i \mid \rho_i(z) = 0, i \in I\}$$

and

$$S_I^{\epsilon_0} = \{z \in \partial \bigcup_{i=1}^{k} D_i \mid \rho_i(z) = -\epsilon_0,\ i \in I\},$$

and each $S_I$ and $S_I^{\epsilon_0}$ is given the natural induced orientation. Then we have

$$\partial R_I^0 = \sum_{j=1}^{k} R_{Ij}^0 + S_I - S_I^{\epsilon_0},$$

$$\begin{aligned}\partial\Big(\sum_I (-1)^{|I|}(R_I^0 \times \triangle_{0I})\Big) &= \sum_I R_I^0 \times \triangle_I - R^0 \times \triangle_0 \\ &\quad + \sum_I (-1)^{|I|} S_I \times \triangle_{0I} - \sum_I (-1)^{|I|} S_I^{\epsilon_0} \times \triangle_{0I},\end{aligned}$$

where the summation is over all ordered increasing subsets of $\{1, \cdots, k\}$.

For any smooth forms $\alpha \in C^\infty_{(0,q)}(\overline{D})$, since the boundary of $D$ is Lipschitz, there exists a componentwise extension $E\alpha \in C^\infty_{(0,q)}(\mathbb{C}^n)$, and $E\alpha$ vanishes outside a neighborhood of $\overline{D}$ (see Stein [35]). We shall assume that $E\alpha$ is supported in $\bigcap_{i=1}^k \{z \in \mathbb{C}^n \mid \rho_i(z) \geq -\epsilon_0\}$ for some small $\epsilon_0 > 0$. Setting $g = \overline{\partial} E\alpha - E\overline{\partial}\alpha$, we get $g = 0$ on $D$, and $g$ vanishes outside a neighborhood of $\overline{D}$. We first assume $\alpha$ (and thus $g$) vanishes on $\mathbb{C}^n \setminus \Omega$. For $z \in D$, using similar arguments as before (applying Stokes' theorem on the 2n-chain $\sum_I (-1)^{|I|}(R_I^0 \times \triangle_{0I})$, we have

$$\begin{aligned}(2.2)\qquad & -\int_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} g \wedge \overline{\partial}_z \Omega^0_{n,q-1} + \int_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} \overline{\partial}_\zeta g \wedge \Omega^0_{n,q} \\ &= \int_{\sum_I R_I^0 \times \triangle_I} g \wedge \Omega^0_{n,q} - \int_{R^0 \times \triangle_0} g \wedge \Omega^0_{n,q}.\end{aligned}$$

We note that since $g \wedge \Omega^0_{n,q}$ is a form of degree $(n, q+1)$ in $\zeta$ variables and

$$\dim_R R_I^0 = 2n - \mid I \mid + 1,$$

we have

$$(2.3)\qquad \int_{R_I^0 \times \triangle_I} g \wedge \Omega^0_{n,q} = 0 \ \text{ if } \ q + 1 \leq n - \mid I \mid .$$

Using the same arguments as in (1.6), we have for $z \in D$,

$$\begin{aligned}(2.4)\qquad & -\int_{R^0 \times \triangle_0} g \wedge \Omega^0_{n,q} \\ &= \overline{\partial}_z \int_{\mathbb{C}^n \times \triangle_0} E\alpha \wedge \Omega^0_{n,q-1} + \int_{\mathbb{C}^n \times \triangle_0} E\overline{\partial}\alpha \wedge \Omega^0_{n,q} + C_n^{-1}\alpha(z).\end{aligned}$$

Substituting (2.2) into (2.4) and using (2.3), we have for $z \in D,\ q+1 \le n-k$,

$$C_n^{-1}\alpha(z) = -\overline{\partial}_z \int\limits_{\mathbb{C}^n \times \triangle_0} E\alpha \wedge \Omega^0_{n,q-1} - \int\limits_{\mathbb{C}^n \times \triangle_0} E\overline{\partial}\alpha \wedge \Omega^0_{n,q}$$

$$- \int\limits_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} g \wedge \overline{\partial}_z \Omega^0_{n,q-1} + \int\limits_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} \overline{\partial}_\zeta g \wedge \Omega^0_{n,q}. \tag{2.5}$$

Define

$$S_q^{(m)}\alpha = C_n \Big\{ \int\limits_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} g \wedge \Omega^0_{n,q-1} - \int\limits_{\mathbb{C}^n \times \triangle_0} E\alpha \wedge \Omega^0_{n,q-1} \Big\} \tag{2.6}$$

for $1 \le q \le n-k-1$.

We have, for $z \in D$,

$$\alpha = S_1^{(m)} \overline{\partial}\alpha, \quad \alpha \in C^\infty_{(0,0)}(\overline{D}), \tag{2.7}$$

and

$$\alpha = \overline{\partial} S_q^{(m)}\alpha + S_{q+1}^{(m)}\overline{\partial}\alpha, \quad \alpha \in C^\infty_{(0,q)}(\overline{D}) \quad \text{and } 1 \le q \le n-k-1. \tag{2.8}$$

Thus we have derived the homotopy formula on $D$ in the case when $\alpha$ vanishes on $\mathbb{C}^n \setminus \Omega$. For the general case we need to modify the above construction. Let $A^\infty(\overline{D}) = C^\infty(\overline{D}) \cap \mathcal{O}(D)$, where $\mathcal{O}(D)$ is the set of holomorphic functions in $D$.

**Theorem 2.** *Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with piecewise $C^\infty$ smooth boundary. Let $D_i \subset\subset \Omega$, $i = 1, \cdots, k$, be a pseudoconvex domain with $C^2$ boundary and $D = \Omega \setminus \overline{(\bigcup_{i=1}^k D_i)}$. We assume that the $\{D_i\}_{i=1}^k$ intersect transversally. For $1 \le q \le n-k$ and every nonnegative integer $m$ there exist linear operators $\tilde{S}_q^{(m)} : C^\infty_{(0,q)}(\overline{D}) \longrightarrow C^m_{(0,q-1)}(\overline{D})$ such that, for every $\alpha \in C^\infty_{(0,q)}(\overline{D})$, $z \in D$, we have*

$$\alpha = \overline{\partial}\tilde{S}_q^{(m)}\alpha + \tilde{S}_{q+1}^{(m)}\overline{\partial}\alpha, \quad \textit{where } 1 \le q \le n-k-1.$$

*When $q = 0 \le n-k$, there exists an operator $\tilde{S}_0 : C^\infty(\overline{D}) \longrightarrow A^\infty(\overline{D})$ such that for every $\alpha \in C^\infty(\overline{D})$,*

$$\alpha = \tilde{S}_0\alpha + \tilde{S}_1^{(m)}\overline{\partial}\alpha, \quad 0 \le n-k-1,$$

*for any $z \in D$.*

**Corollary 2.1.** *Let $D$ be the same as in Theorem 2. If $\alpha \in C^\infty_{(0,q)}(\overline{D})$ and $\overline{\partial}\alpha = 0$, where $1 \le q \le n-k-1$, then there exists a $u \in C^\infty_{(0,q-1)}(\overline{D})$ satisfying $\overline{\partial}u = \alpha$ in $D$.*

*Proof.* Let $E\alpha$ be a $C^\infty$ extension of $\alpha$ to $\mathbb{C}^n$ such that $E\alpha$ vanishes outside a small neighborhood of $\overline{D}$. Using the construction of the operator $T_q$ in section I, we define

$$\tilde{S}_q^{(m)}\alpha = C_n \Big\{ - \int\limits_{\sum_I (-1)^{|I|} R_I \times \triangle_{0I}} (\overline{\partial}E\alpha - E\overline{\partial}\alpha) \wedge \Omega_{n,q-1}$$

$$+ \int\limits_{\sum_I (-1)^{|I|} R_I^0 \times \triangle_{0I}} (\overline{\partial}E\alpha - E\overline{\partial}\alpha) \wedge \Omega^0_{n,q-1} - \int\limits_{\mathbb{C}^n \times \triangle_0} E\alpha \wedge \Omega^0_{n,q-1} \Big\}$$

for $1 \le q \le n-k$, and

$$\tilde{S}_0\alpha = C_n \sum_I \int_{R_I \times \triangle_I} (\overline{\partial} E\alpha - E\overline{\partial}\alpha) \wedge \Omega_{n,0}.$$

From the above argument and the proof of Theorem 1, we see that the homotopy formulas hold. To see that $S_q^{(m)}\alpha \in C^m_{(0,q-1)}(\overline{D})$, we use (2.1) and the same arguments as in the proof of Theorem 1. Thus for each nonnegative integer $m$, there exists a solution $u_m \in C^m_{(0,q-1)}(\overline{D})$ such that $\overline{\partial} u_m = \alpha$ in $D$. To extract a convergent sequence from $u_m$ in order to obtain a $C^\infty$ solution $u$, we repeat an argument of Kohn (see also Michel-Shaw [26]). We omit the details. □

Next we want to prove the boundary regularity for $\overline{\partial}$ on an annulus between a pseudoconvex domain and an intersection of smooth pseudoconvex domains. For every increasing multi-index $I = (i_1, \cdots, i_\mu), 1 \le i_1 < \cdots < i_\mu \le k$, we define $D_I = \bigcap_{i \in I} D_i$. We set, for fixed $1 \le \gamma \le k$, $1 \le j_\nu \le k$,

$$\Omega_2 = \bigcup_{\nu=1}^{\gamma} D_{Ij_\nu} \quad \text{and} \quad A = \Omega \setminus \Omega_2,$$

where $I = (i_1, \cdots, i_\mu),\ \gamma + \mu \le k,\ $ and $\ j_\nu \notin I$. We set $D_\phi = \mathbb{C}^n$ and assume $\Omega_2$ and $A$ are connected. We also assume that each $\Omega \setminus \{\bigcup_{\nu=1}^{\mu+\gamma-\tilde{\mu}} (D_1 \cap \cdots \cap D_{\tilde{\mu}}) \cap D_{\tilde{\mu}+\nu}\}$ is connected for each $0 \le \tilde{\mu} \le \mu$. We first prove the regularity for $\overline{\partial}$ on $A$.

**Theorem 3.** *For every $f \in C^\infty_{(0,q)}(\overline{A})$, where $1 \le q \le n-1-\gamma$, such that $\overline{\partial} f = 0$ in $A$, there exists a $g \in C^\infty_{(0,q-1)}(\overline{A})$ satisfying $\overline{\partial} g = f$ in $A$.*

**Corollary 3.1.** *For every $\alpha \in C^\infty_{(0,q)}(\mathbb{C}^n)$ such that $\overline{\partial}\alpha = 0$ in $\mathbb{C}^n$ and* $\operatorname{supp}\alpha \subset \overline{\Omega}_2$, *where $1 \le q \le n-\gamma$, there exists a $u \in C^\infty_{(0,q-1)}(\mathbb{C}^n)$ satisfying $\overline{\partial} u = \alpha$ in $\mathbb{C}^n$ and* $\operatorname{supp} u \subset \overline{\Omega}_2$.

In particular, we have the following important case when $| I |= k-1$ and $\gamma = 1$. Note that $D_I = \bigcap_{i=1}^k D_i$ if $| I |= k$.

**Theorem 3′.** *Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with piecewise smooth $C^\infty$ boundary. Let $D_i \subset\subset \Omega, i = 1, \cdots, k$, be pseudoconvex domains with $C^2$ boundary, and $G = \Omega \setminus (\bigcap_{i=1}^k D_i)$. For every $\overline{\partial}$-closed $f \in C^\infty_{(0,q)}(\overline{G}), 1 \le q \le n-2$, there exists a $u \in C^\infty_{(0,q-1)}(\overline{G})$ such that $\overline{\partial} u = f$ in $G$.*

Let $D^0 = \bigcap_{i=1}^k D_i$. Theorem 3′ implies the following:

**Corollary 3.1′.** *For every $\alpha \in C^\infty_{(0,q)}(\mathbb{C}^n)$ such that $\overline{\partial}\alpha = 0$ in $\mathbb{C}^n$ and* $\operatorname{supp}\alpha \subset$ $\overline{D}^0$, *where $1 \le q \le n-1$, there exists a $u \in C^\infty_{(0,q-1)}(\mathbb{C}^n)$ satisfying $\overline{\partial} u = \alpha$ in $\mathbb{C}^n$ and* $\operatorname{supp} u \subset \overline{D}^0$.

To prove Theorem 3, we need the following lemma.

**Lemma 3.2.** *Let $I = (i_1, \cdots, i_\mu)$ and $0 < \gamma \le n-1,\ \mu + \gamma \le n-1,\ \mu, \gamma \le k$. If $1 \le q \le n-1-\gamma$, and if $f \in C^\infty_{(0,q)}(\overline{A})$ is such that $\overline{\partial} f = 0$ in $A$ and $f = 0$ in $\Omega \setminus D_I$, there exists a $g \in C^\infty_{(0,q-1)}(\overline{A})$ such that $\overline{\partial} g = f$ in $A$ and $g = 0$ in $\Omega \setminus D_I$.*

*Proof.* We shall use induction on $\mu$ for all $0 < \gamma \le n-1-\mu$. For $\mu = 0$, this is proved in Theorem 2 (since $1 \le q \le n-1-\gamma$ and $D_\phi = \mathbb{C}^n$). We assume that the lemma holds for $\mu - 1$, $\mu \ge 1$ and all admissible $\gamma$, $q$.

It suffices to prove the assertion for $I = (1, \cdots, \mu)$, $\mu \ge 1$, $\mu + \gamma \le n-1$ and $0 < \gamma \le n-1$. Let $f \in C^\infty_{(0,q)}(\overline{A})$, $1 \le q \le n-1-\gamma$, and $f = 0$ on $\Omega \setminus D_I$. We define $J = (1, \cdots, \mu - 1)$ if $\mu \ge 2$ and $J = \varnothing$ if $\mu = 1$. Let

$$A' = \Omega \setminus \bigcup_{\nu=1}^{\gamma} D_{Jj_\nu}.$$

Then $\overline{\partial} f = 0$ in $A'$ and $f = 0$ on $\Omega \setminus D_J$. By induction there exists a $g' \in C^\infty_{(0,q-1)}(\overline{A}')$ such that

$$\overline{\partial} g' = f \quad \text{in } A'$$

and

$$g' = 0 \quad \text{in } \Omega \setminus D_J.$$

Setting $A'' = \Omega \setminus \{D_I \cup (\bigcup_{\nu=1}^{\gamma} D_{Jj_\nu})\}$, one has $\overline{\partial} g' = f = 0$ in $A''$ and $g' = 0$ on $\Omega \setminus D_J$. If $q \ge 2$, since $((q-1)+\gamma+1) = q+\gamma \le n-1$, we can again apply induction to the set

$$D_I \cup \left( \bigcup_{\nu=1}^{\gamma} D_{Jj_\nu} \right) = D_{J\mu} \cup \left( \bigcup_{\nu=1}^{\gamma} D_{Jj_\nu} \right).$$

Thus there exists an $h' \in C^\infty_{(0,q-2)}(\overline{A}'')$ such that

$$\overline{\partial} h' = g' \text{ in } A''$$

and

$$h' = 0 \text{ in } \overline{\Omega \setminus D_J}.$$

Extending $h'$ smoothly into $\overline{A}'$ and denoting the extension by $Eh'$, we set $\widetilde{g} = g' - \overline{\partial} Eh'$. Then $\widetilde{g} \in C^\infty_{(0,q-1)}(\overline{A}')$, $\overline{\partial}\widetilde{g} = f$ in $A'$, and $\widetilde{g} = 0$ in $\overline{A}''$. We extend $\widetilde{g}$ to $\overline{A}$, as $g$ by setting $g = \widetilde{g}$ in $A'$ and $g = 0$ in $\overline{A} \setminus A'$. Since $f = 0$ in $\overline{A} \setminus A'$, we have $g \in C^\infty_{(0,q-1)}(\overline{A})$, $\overline{\partial} g = f$ in $A$ and $g = 0$ in $\Omega \setminus D_I$. Thus the lemma is proved when $q = 2$.

When $q = 1$, $g'$ is holomorphic on $\Omega \setminus D_J$. By Hartogs' theorem, there exists a holomorphic function $h$ such that $\overline{\partial} h = 0$ in $\Omega$ and $h = g'$ on $\Omega \setminus D_J$. Setting $\widetilde{g} = g' - h$ in $A'$, we have $\overline{\partial}\widetilde{g} = f$ in $A'$ and $\widetilde{g} = 0$ in $A''$. Repeating the arguments as before, the lemma is proved for $q = 1$ also. □

*Proof of Theorem 3.* We argue by induction on $| I |= \mu$.

When $\mu = 0$, this again was proved in Theorem 2. For $\mu \ge 1$, if $\overline{\partial} f = 0$ in $A$, we have $\overline{\partial} f = 0$ on $\Omega \setminus D_I$. Thus writing $D_I = D_{I'j_\mu}$ where $I' = (i_1, \cdots, i_{\mu-1})$, we can use induction (since $q \le n-2$) to find a $V \in C^\infty_{(0,q-1)}(\overline{\Omega \setminus D_I})$ such that $\overline{\partial} V = f$ on $\Omega \setminus D_I$. Extending $V$ smoothly into $\widetilde{V}$ in $\Omega$, we define $f_0 \equiv f - \overline{\partial}\widetilde{V}$. Then $\overline{\partial} f_0 = 0$ in $A$ and $f_0 = 0$ on $\Omega \setminus D_I$. Thus we can apply Lemma 3.2 to find a solution $g$ in $A$, and Theorem 3 is proved. □

Corollary 3.1 follows easily, since $\alpha = \overline{\partial} U$ in some large ball $B$ containing each $D_i$. By Theorem 3, there exists a $V \in C^\infty_{(0,q-1)}(\overline{B \setminus \Omega_2})$ such that $\overline{\partial} V = U$. Extending $V$ smoothly into $\widetilde{V}$ on $B$ and setting $u = U - \overline{\partial}\widetilde{V}$, we have proved the corollary when $2 \le q \le n - \gamma$. When $q = 1$, it follows easily from Hartogs' Theorem. □

## III. Applications to the local solvability of $\overline{\partial}_b$

Let $M$ be the boundary of a bounded smooth pseudoconvex domain $\Omega$ in $\mathbb{C}^n$. Let $\omega$ be a connected open subset of $M$ with piecewise smooth boundary $\partial\omega$. By this we mean there exist bounded domains $D_i$, $i = 1, \ldots, k$, with smooth boundary $\partial D_i$, such that

$$\omega = M \cap \left( \bigcap_{i=1}^{k} D_i \right),$$

where $\partial D_i$ and $M$ intersect transversally wherever they intersect.

**Definition.** $\omega$ is called a *domain with admissible boundary* $\partial\omega$ if the following conditions hold:

i) $\Omega_I \equiv \Omega \cap (\bigcap_{i=1}^{k} D_i)$ is a bounded piecewise smooth pseudoconvex domain (as defined in section I).

ii) For each $1 \le i \le k$, the set $\Omega_i^c \equiv \Omega \cap (\mathbb{C}^n \setminus \overline{D}_i)$ is equal to $\Omega$ intersected with a bounded smooth pseudoconvex domain.

We note that i) and ii) imply that $\partial\omega$ consists of smooth pieces which lie in Levi-flat hypersurfaces. Examples of admissible boundaries are those defined by real hyperplanes in $\mathbb{C}^n$.

**Theorem 4.** *Let $\omega \subset M$ be a domain with admissible boundary. For every $\overline{\partial}_b$-closed form $\alpha \in C^\infty_{(0,q)}(\overline{\omega})$, $1 \le q \le n - k - 2$, there exists a $u \in C^\infty_{(0,q-1)}(\overline{\omega})$ such that*

$$\overline{\partial}_b u = \alpha \quad \textit{in } \omega. \tag{4.1}$$

*Proof.* Let $\tilde{\alpha}$ be a $C^\infty$ extension of $\alpha$ to $\overline{\Omega}_I$ such that $\tilde{\alpha} \in C^\infty_{(0,q)}(\overline{\Omega}_I)$ and $\overline{\partial}\tilde{\alpha}$ vanishes to infinite order on $\omega$. Let

$$\alpha_1 = \overline{\partial}\tilde{\alpha} \quad \text{for} \quad z \in \overline{\Omega}_I,$$

and let $\tilde{\alpha}_1$ be a $C^\infty$ extension of $\alpha_1$ to $\overline{\Omega}$ such that $\overline{\partial}\tilde{\alpha}_1$ vanishes to infinite order on $\partial\Omega$ and

$$\overline{\partial}\tilde{\alpha}_1 = 0 \quad \text{on} \quad \Omega_I.$$

Let

$$\alpha_2 = \overline{\partial}\tilde{\alpha}_1 \quad \text{for } z \in \overline{\Omega}$$

and extend $\alpha_2$ to be zero outside $\overline{\Omega}$. We have

$$\overline{\partial}\alpha_2 = 0 \quad \text{in} \quad \mathbb{C}^n$$

and

$$\text{supp } \alpha_2 \ \subset \ \overline{\Omega}_I^c$$

where $\Omega_I^c = \Omega \setminus \overline{\Omega}_I$. Thanks to the assumption ii) and the equality $\Omega_I^c = \bigcup_{i=1}^k \Omega_i^c$, we can apply Corollary 3.1 (when $\mu = 1, \gamma = k$) since $q + 2 \leq n - k$. Thus there exists a $\beta_0 \in C^\infty_{(0,q+1)}(\mathbb{C}^n)$ such that

$$\overline{\partial}\beta_0 = \alpha_2 \quad \text{in} \;\; \mathbb{C}^n$$

and

$$\text{supp } \beta_0 \;\subset\; \overline{\Omega}_I^c.$$

Setting $\alpha_1' = \tilde{\alpha}_1 - \beta_0$ in $\Omega$, we have that $\alpha_1'$ is a $\overline{\partial}$-closed extension of $\alpha_1$ from $\Omega_I$ to $\Omega$ and $\alpha_1'$ vanishes to infinite order on $\partial\Omega$. Extending $\alpha_1'$ to be zero outside $\Omega$ and using Corollary 3.1 again, we see that there exists a $V_0 \in C^\infty_{(0,q)}(\mathbb{C}^n)$ such that

$$\overline{\partial}V_0 = \alpha_1' \quad \text{in} \;\; \mathbb{C}^n$$

and

$$\text{supp } V_0 \subset \overline{\Omega}.$$

Let

$$\alpha' = \tilde{\alpha} - V_0 \quad \text{in } \overline{\Omega}_I.$$

We have $\alpha' \in C^\infty_{(0,q)}(\overline{\Omega}_I)$ and

$$\overline{\partial}\alpha' = 0 \quad \text{in} \;\; \Omega_I,$$
$$\alpha' = \alpha \quad \text{on} \;\; \omega.$$

By Corollary 1.1 and the assumption i) for admissible domains, there exists a $U' \in C^\infty_{(0,q-1)}(\overline{\Omega}_I)$ such that

$$\overline{\partial}U' = \alpha' \quad \text{in} \;\; \Omega_I.$$

Restricting $U'$ to $\omega$ and denoting it by $u$, we have $\overline{\partial}u = \alpha$ in $\omega$ and $u \in \mathcal{C}^\infty_{(0,q-1)}(\overline{\omega})$. This proves Theorem 4. □

*Remark.* We note that the condition on the degree $q$ and $k$ cannot be relaxed. Let $M$ be the unit sphere in $\mathbb{C}^n$, $n \geq 3$, and $M = \{z|\ |z_1|^2 + \cdots + |z_n|^2 = 1\}$. Let

$$\omega = M \cap \left( \bigcap_{i=3}^{i=n} \{z|\ |z_i|^2 < \frac{1}{2(n-2)}\} \right).$$

Then $\omega$ is a domain with admissible boundary with $k = n-2$. We shall show that Equation (4.1) is not solvable for $q = 1$ in $\omega$. Let

$$\alpha = \frac{\overline{z}_1 d\overline{z}_2 - \overline{z}_2 d\overline{z}_1}{(|z_1|^2 + |z_2|^2)^2}.$$

Then $\alpha \in C^\infty_{0,1}(\overline{\omega})$ and $\overline{\partial}_b\alpha = 0$ in $\omega$. Let $M_0 = M \cap \{z_3 = \frac{1}{\sqrt{2(n-2)}}, \cdots, z_n = \frac{1}{\sqrt{2(n-2)}}\}$. If $\alpha = \overline{\partial}_b u$ for some $u \in C^\infty(\overline{\omega})$, then we have

$$\int_{M_0} \alpha dz_1 \wedge dz_2 = \int_{M_0} \overline{\partial}u \wedge dz_1 \wedge dz_2 = 0.$$

On the other hand,

$$\begin{aligned}\int_{M_0} \alpha dz_1 \wedge dz_2 &= \frac{1}{4}\int_{|z_1|^2+|z_2|^2=\frac{1}{2}} \{\overline{z}_1 d\overline{z}_2 - \overline{z}_2 d\overline{z}_1\} dz_1 \wedge dz_2 \\ &= \frac{1}{2}\int_{|z_1|^2+|z_2|^2<\frac{1}{2}} d\overline{z}_1 \wedge d\overline{z}_2 \wedge dz_1 \wedge dz_2 \\ &\neq 0.\end{aligned}$$

Thus the condition on the number of intersection $k$ cannot be removed.

On the other hand, if we take

$$\omega = M \cap (\bigcap_{i=4}^{i=n} \{z|\ |z_i|^2 < \frac{1}{2(n-2)}\}),$$

where $n > 4$, then using Theorem 4 we can find a $u \in C^\infty(\overline{\omega})$ satisfying $\overline{\partial}_b u = \alpha$ in $\omega$.

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